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TRANSFER IN GALOIS COHOMOLOGY COMMUTES WITH TRANSFER IN THE MILNOR RING

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The purpose of this paper is to study the relation between transfer maps defined in the theory of quadratic forms, the algebraic K-theory of Milnor and Galois cohomology, respectively. The results should be viewed as an addendum to [8] and to an unpublished letter of John Tate.

For a field F, $\mathrm{ch}(F)\neq 2$, we have an associated ring, $\hat{W}(F)$, called the Witt–Grothendieck ring of regular quadratic forms over F [4]. If E is a finite extension of F then the trace linear functional induces a transfer map, $T\colon \hat{W}(E)\to \hat{W}(F)$, which is an additive group homomorphism [4, Chapter 7]. Now consider $K_*(F)$, the Milnor ring of F. For any finite separable extension $E=F(\alpha)$ of F we have an induced map $N_\alpha : K_*(E)\to K_*(F)$ defined by Bass and Tate in [1]. It is as yet unknown whether N_α is independent of the choice of primitive element α [1]. If $k_*(F)=K_*(F)/2K_*(F)$, the reduced Milnor ring, then we will show that if E is Galois over F of odd degree then $N_\alpha : k_*(E)\to k_*(F)$ is indeed independent of α. In fact we will show that the restriction of N_α to $k_2(E)$ is the canonical transfer map (mod 2) defined by Milnor in [7].

Let $H^*(F)=H^*(F;Z/2Z)$ be the Galois cohomology ring of the absolute Galois group of F with coefficients in $Z/2Z$. If $\mathrm{Gal}(\bar{F}/F)$ is the Galois group of a separable closure $\bar{F}$ over F then in terms of group cohomology

$$H^*(\mathrm{Gal}(\bar{F}/F))=\varinjlim_{i\in\Lambda} H^*(\mathrm{Gal}(F_i/F))$$

where $\{F_i\}_{i\in\Lambda}$ is the set of all finite Galois extensions of F contained in $\bar{F}$ ([9]). In this case corestriction gives us a canonical transfer map, $\mathrm{cores}: H^*(E)\to H^*(F)$ whenever E is a finite extension of F.

There are two main results in this paper. The first generalizes a result of Tate which appears in a letter to S. Rosset. For any field F we have a ring homomorphism $h_F\colon k_*(F)\to H^*(F)$ [6; Lemma 6.1]. If E is a Galois extension of F of odd degree, $E=F(\alpha)$, let $N=N_\alpha$ be the canonical transfer map.

[1] K. Kato has recently proved the independence of the choice of α (unpublished manuscript).

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Theorem 1. *For any $n \geq 0$*

$$\begin{array}{ccc} k_n(E) & \xrightarrow{h_E} & H^n(E) \\ \downarrow N & & \downarrow \text{cores} \\ k_n(F) & \xrightarrow{h_F} & H^n(F) \end{array}$$

is a commutative diagram of abelian groups.

If we assume that for any Galois extension $E=F(\alpha)$ the transfer map N_α is independent of α and is also functorial with respect to a tower of extensions then we could drop the 'odd degree' assumption in Theorem 1. Tate proved a slightly different result which covers the case $n=2$ with N being the above cited transfer map of Milnor. The cases $n=0$, 1 are trivial.

The second theorem is a corollary to the main theorem of [8]. Let w_i, $i=0,1,2,3,\dots$, be the Stiefel–Whitney invariants defined by Delzant–Milnor and let us view them as maps $w_n: \hat{W}(F) \to H^n(F)$. $\hat{I}(F)$ will denote the augmentation ideal, i.e. the kernel of the dimension homomorphism $\dim: \hat{W}(F) \to Z$. If $t=2^{n-1}$ then $w_t: \hat{I}^n(F) \to H^t(F)$ is a group homomorphism.

Theorem 2.

$$\begin{array}{ccc} \hat{I}^n(E) & \xrightarrow{w_t} & H^t(E) \\ \downarrow T & & \downarrow \text{cores} \\ \hat{I}^n(F) & \xrightarrow{w_t} & H^t(F) \end{array}$$

is a commutative diagram of abelian groups. $\hat{I}^n(F)$ is the n-th power ideal of $\hat{I}(F)$ and T is the transfer map restricted to $\hat{I}^n(E) \leq \hat{W}(E)$.

1.

For any field F, $\mathrm{ch}(F) \neq p$, Tate has constructed an associated field F' which will play an important role in what follows. The construction is introduced in [10] but in spirit it goes back to the methods used in [1, Section 5]. For this paper we are basically interested in the case $p=2$.

Consider the set of all subextensions of a separable closure, $\bar{F}$, of F which (1) contain F and (2) can be realized as a set-theoretic union of finite prime-to-p extensions of F. A Zorn's lemma argument implies that this set contains a maximal element. Call it F'. This field F' has some very interesting and useful properties. These properties are asserted in Tate's letter. Since the letter remains unpublished we will *sketch* the proofs below. I would like to thank Jack Sonn and Roger Ware for conversations which helped me better understand the nature of F'.

Property 1. *If E is any finite extension of F contained in F' then the degree $[E:F]$ is prime to p.*

Proof. $E=F(\alpha_1,\dots,\alpha_r)$ where $\alpha_i\in F'$. Hence their irreducible polynomials over F must be of degree prime to p.

Property 2. *If E is any finite extension of F' contained in $\bar{F}$ then $[E:F']$ is equal to p^a for some nonnegative integer a.*

Proof. This will follow if we show that for any $\alpha\in E$ the irreducible polynomial of α over F', $\mathrm{Irr}(\alpha,F')$, has as its degree a power of p. All the coefficients of $\mathrm{Irr}(\alpha,F')$ lie in finite prime-to-p extensions of F. If the degree of α over F' is also prime-to-p a straightforward argument shows that $F'(\alpha)$, a proper extension of F', contradicts the maximality of F'. Assume that the degree of α over F' is $p^a n$, $a\geq 1$, $n\neq 0$ and $(p,n)=1$. Let E_{N} be the normal closure of E in $\bar{F}$. It is a Galois extension of F' whose degree over F' is divisible by $p^a n$. By the above assumptions, a Sylow p-subgroup, $S(p)$, of $\mathrm{Gal}(E_{\mathrm{N}}/F')$ is a nontrivial proper subgroup. Therefore the fixed subfield, $E_{\mathrm{N}}^{S(p)}$, would be a nontrivial prime-to-p extension of F' which we have already seen cannot happen. It follows that $n=1$ and we are done.

Property 3. *Let E be a finite extension of F' contained in $\bar{F}$, $[E:F']=p^a$. Then there exists a tower of fields*

$$F'=E_1\subset E_2\subset\cdots\subset E_a=E$$

such that $[E_i:E_{i-1}]=p$.

Proof (by induction on a). To establish the property we need only find a subfield K of E whose degree over F' is p^{a-1}. $G=\mathrm{Gal}(\bar{F}/F')$ is a pro-p-group since by Property 2 all finite extensions of F' contained in $\bar{F}$ are p-power extensions. Galois theory implies that E is the fixed subfield of a subgroup H of G where $[G:H]=p^a$. We will find a subgroup H_1, $H\leq H_1\leq G$ so that $[G:H_1]=p^{a-1}$. Letting $K=\bar{F}^{H_1}$ we will get the desired subfield K.

H is an open subgroup of G (finite index). By the class equation it also follows that H has only a finite number of conjugates in G. Let $H'=\bigcap_{x\in G}x^{-1}Hx$, then H' is an open normal subgroup of G containing H. G/H' is a finite p-group containing H/H' as a subgroup. By the Sylow theorems we can find H_1, normal in G, with $H'\leq H_1\leq H$ and $[G:H_1]=p^{a-1}$. This then completes the proof.

Let us note that from the above proof it follows that if we assume E/F is Galois we can construct the E_i so that each is Galois over E_{i-1}.

Property 4. *Let $p=2$. Then the restriction map* res $: H^*(F) \to H^*(F')$ *is injective.*

Proof. Since coefficients are in $Z/2Z$ it is well known that for finite groups $G' \leq G$, res $: H^*(G) \to H^*(G')$ is injective if the index of G' in G is odd (prime to 2). By Property 1 every finite extension of F in F' is odd degree. Using the definition of Galois cohomology as a direct limit of finite group cohomology, injectivity for res $: H^*(F) \to H^*(F')$ follows.

2.

Following the notation of [6] $K_n(F)$ will denote the Milnor K-group of n-fold symbols over F. $K_n(F)$ is additively generated by symbols $l(a_1) \cdots l(a_n)$ where $a_i \in F^{\cdot}$ is a unit in F. The quotient group $K_n(F)/2K_n(F)$ will be denoted by $k_n(F)$. $k_*(F)$ is the reduced Milnor ring and as a group is isomorphic to $\coprod_{n \geq 0} k_n(F)$. By $k_\pi(F)$ we mean the $Z/2Z$ algebra consisting of all formal series $\gamma_0 + \gamma_1 + \gamma_2 + \cdots$ with $\gamma_i \in k_i(F)$.

Suppose E is a finite separable extension of F with primitive element α, i.e. $E = F(\alpha)$. Bass and Tate define a transfer map $N_\alpha : K_n(E) \to K_n(F)$. By reducing mod $2K_n(E)$ we have $N_\alpha : k_n(E) \to k_n(F)$. The question is whether N_α is independent of α and if so is it also functorial. By functorial we mean that if $E \supset K \supset F$ is a tower of separable extensions with $K = F(\alpha)$, $E = K(\beta)$ and $E = F(\gamma)$ then $N_\gamma = N_\alpha \circ N_\beta$. A partial solution is suggested in [1] but to my knowledge the question still remains open. What we will now show is that we can answer the first question in the affirmative if E is Galois over F of odd degree.

Proposition 1. *Let E be a finite Galois extension of F of odd degree and suppose $E = F(\alpha) = F(\beta)$. Then $N_\alpha = N_\beta : k_*(E) \to k_*(F)$.*

Proof. The proof is essentially an application of the partial solution suggested by Bass and Tate and so we will refer the reader to [1, pp. 39–40]. For the field L in [1] substitute the field F' constructed in the previous section. Since F' is a union of odd-degree extensions the restriction map $j : k_*(F) \to k_*(F')$ is injective by [1, p. 40]. By the standard theorems of Galois theory (see for example [5]) $[F'(\alpha_i) : F']$ divides $[F(\alpha) : F]$ for all i. Since $[F(\alpha) : F]$ is odd and since F' has only 2-power degree extensions it follows that for all i, $F'(\alpha_i) = F'$. The general case now follows from the case of degree 1 extensions where $N_\alpha \equiv$ identity.

3.

For a field F, ch$(F) \neq 2$, with separable closure $\bar{F}$ let G be the absolute Galois group of F, i.e. $G = \mathrm{Gal}(\bar{F}/F)$. By [6, Section 6] we have a group homomorphism

$\delta: F^{\cdot}/(F^{\cdot})^2 \to H^1(G)$, where $F^{\cdot}$ is the multiplicative group of units in the field F. By viewing $k_1(F)$ as $F^{\cdot}/(F^{\cdot})^2$ and by using the cup product in cohomology we get an induced ring homomorphism $h_F: k_*(F) \to H^*(G)$. For each $n \geq 0$, $h_F: k_n(F) \to H^n(G)$ is a group homomorphism. Following the usual notation we will denote $H^*(G)$ by $H^*(F)$ (F completely determines the isomorphism class of G). There are many examples for which h_F is known to be bijective, e.g. finite, local, global or real closed fields [6, Lemma 6.2].

As we noted previously (Proposition 1), if E is a Galois extension of F of odd degree then we have a canonical transfer map $N: k_n(E) \to k_n(F)$ for any $n \geq 0$.

Theorem 1. *For E and F as above we have the following commutative diagram of groups and group homomorphisms:*

$$\begin{array}{ccc} k_n(E) & \xrightarrow{h_E} & H^n(E) \\ \big\downarrow N & & \big\downarrow \text{cores} \\ k_n(F) & \xrightarrow{h_F} & H^n(F) \end{array} \tag{1}$$

where cores is the corestriction map in cohomology.

Remark. A stronger result for the case $n=2$ is proved by Tate in [10]. Our proof will be an application of Tate's methods to our particular situation. The idea is to create a cube whose vertices are groups and whose edges are group homomorphisms in such a way that diagram (1) is one of the faces. By a sequence of lemmas we will show that all the other faces commute. This in turn will imply that (1) is a commutative square.

In proving the lemmas we will only assume that E is a finite separable extension of F. For the theorem itself we will need both the assumptions that E is Galois and of odd degree over F. We have $E = F(\alpha)$ for some $\alpha \in E$ and a possibly noncanonical transfer map $N = N_\alpha: k_n(E) \to k_n(F)$ (see Section 2). Let L be an extension of F contained in $\bar{F}$. Note that although L is arbitrary we will eventually apply the lemmas to the case where $L = F'$, the special extension of F constructed in Section 1. Let $\tilde{L}$ be defined as $E \otimes_F L$. $\tilde{L}$ can be realized as a direct product of fields, $\tilde{L} \cong \prod_{i=1}^r L_i$, where each L_i is a finite separable extension of L. If we assume that E is Galois over F then each L_i is Galois over L and furthermore $[L_i : L]$ divides $[E:F]$. If we let $p(t) = \operatorname{Irr}(\alpha, F)$ then in $L[t]$ $p(t)$ will decompose into a product $p_1(t) \cdots p_r(t)$ where each $p_i(t)$ is a monic irreducible polynomial. We can take as L_i the quotient $L[t]/(p_i(t))$ where $(p_i(t))$ represents the principal ideal generated by $p_i(t)$. In general $\tilde{L}$ is not a field (unless $p(t)$ is irreducible in $L[t]$) and so we need to make some definitions. For $n \geq 0$ we will define $k_n(\tilde{L})$ as a direct product $\prod_{i=1}^r k_n(L_i)$ and $H^n(\tilde{L})$ as $\prod_{i=1}^r H^n(L_i)$. When we refer to a restriction, corestriction or transfer map we will mean the appropriate sum or direct product maps. For example $N: k_n(\tilde{L}) \to k_n(L)$ is the sum in $k_n(L)$ of the maps $N_{\alpha_i}: k_n(L_i) \to k_n(L)$ where α_i is a root of $p_i(t)$ in L_i. By $\mathrm{res}: k_n(E) \to k_n(\tilde{L})$ we mean the direct product map of the $\mathrm{res}_i: k_n(E) \to k_n(L_i)$.

Lemma 1.

$$\begin{tikzcd} k_n(E) \arrow[r, "\mathrm{res}"] \arrow[d, "N"] & k_n(\tilde{L}) \arrow[d, "N"] \\ k_n(F) \arrow[r, "\mathrm{res}"] & k_n(L) \end{tikzcd}$$

is a commutative diagram of groups and group homomorphisms.

Proof. The diagram is one portion of the commutative exact sequence diagram of [1, Section 5.8].

Lemma 2. *Let* $\mathrm{res}: H^n(E) \to H^n(\tilde{L})$ *be the direct product map* $\prod_{i=1}^r \mathrm{res}_i$ *where* $\mathrm{res}_i: H^n(E) \to H^n(L_i)$. *Let* $\mathrm{cores}: H^n(\tilde{L}) \to H^n(L)$ *be the sum in* $H^n(L)$ *of the maps* $\mathrm{cores}_i: H^n(L_i) \to H^n(L)$. *Then the following diagram of groups and group homomorphisms is commutative:*

$$\begin{tikzcd} H^n(E) \arrow[r, "\mathrm{res}"] \arrow[d, "\mathrm{cores}"'] & H^n(\tilde{L}) \arrow[d, "\mathrm{cores}"] \\ H^n(F) \arrow[r, "\mathrm{res}"] & H^n(L) \end{tikzcd}$$

Proof. Let $G_F = \mathrm{Gal}(\bar{F}/F)$ and let us use the notation of group cohomology to write $H^n(F) = H^n(G_F)$. Let $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_s$ be the roots of $p(t) = \mathrm{Irr}(\alpha, F)$ in $\bar{F}$. In $L[t]$ $p(t) = p_1(t) \cdots p_r(t)$. Choose the ordering so that $\alpha = \alpha_1$ is a root of $p_1(t)$. Then the composite EL of E and L in $\bar{F}$ is isomorphic to $L_1 \cong L[t]/(p_1(t))$. By separability all the α_i are distinct.

Partition the roots into r disjoint subsets $S_i = \{\alpha_{i1}, \ldots, \alpha_{is_i}\}$, $1 \le i \le r$, where S_i is the set of roots of $p_i(t)$ in $\bar{F}$. In particular we have $\sum_{i=1}^r s_i = s$. We may order S_1 so that $\alpha_{11} = \alpha$. For each $1 \le i \le r$ there exists an embedding η_i of E in $\bar{F}$ over F such that $\eta_i(\alpha_{11}) = \alpha_{i1}$. Let $\eta_i(E)$ be denoted by E_i. Then $E_i = F(\alpha_{i1})$ and L_i is isomorphic to the composite $E_i L$.

For each i, $1 \le i \le r$, we have the following diagram of fields:

$$\begin{tikzcd} E \arrow[r, "\eta_i"] \arrow[dr, dash] & E_i \arrow[d, dash] \arrow[rr, dash] & & E_i L \cong L_i \arrow[d, dash] \\ & F \arrow[rr, dash] & & L \end{tikzcd}$$

and a corresponding diagram for cohomology

$$\begin{tikzcd}
H^n(G_E) \arrow[r, "\eta_i^*"] \arrow[d, "\mathrm{cores}"'] & H^n(G_{E_i}) \arrow[r, "\mathrm{res}_i"] & H^n(G_{L_i}) \arrow[d, "\mathrm{cores}_i"] \\
H^n(G_F) \arrow[rr, "\mathrm{res}"'] & & H^n(G_L)
\end{tikzcd}$$

η_i induces a map from $G_{E_i} \to G_E$ and η_i^* is the induced map on cohomology. To prove the lemma we need to show that

$$\mathrm{res} \circ \mathrm{cores} = \sum_{i=1}^{r} \mathrm{cores}_i \circ \mathrm{res}_i \circ \eta_i^*. \tag{2}$$

Let us recall the definition of corestriction given in [11]. Given a profinite group G and a subgroup G' of finite index choose a set of representatives for the right cosets of G' in G (a right transversal). If R is the right transversal chosen then $G = \bigcup_{r \in R} G'r$. On the cochain level define

$$\mathrm{cores}(f)(X_1, \dots, X_n) = \sum_{r \in R} f(r^{(0)} X_1 r^{(1)^{-1}}, \dots, r^{(n-1)} X_n r^{(n)^{-1}}),$$

where f is a G' cochain and $(X_1, \dots, X_n) \in G^n$. The $r^{(i)}$ are determined inductively by $r^{(0)} = r$ and for $1 \le i \le n$ $r^{(i)} \in G' r^{(i-1)} X_i$, $r^{(i)} \in R$. As noted in [11] this defines a map on cohomology independent of the choice of R. As it will turn out formula (2) is valid even on the cochain level.

For each i, j, $1 \le i \le r$, $1 \le j \le s_i$, let $\tau_{ij} \in G_L$ be chosen so that $\tau_{ij}(\alpha_{ij}) = \alpha_{i1}$. For fixed i the set $\{\tau_{ij}\}_{j=1}^{s_i}$ gives a right transversal for G_{L_i} in G_L. It follows that $\{\eta_i^{-1} \tau_{ij}\}_{i=1}^{r}{}_{j=1}^{s_i}$ provides a right transversal for G_E in G_F. Let f be an n-cochain representing an element in $H^n(G_E)$. For $(X_1, \dots, X_n) \in G_L^n$ we have

$$\begin{aligned}
&\sum_{i=1}^{r} \mathrm{cores}_i \circ \mathrm{res}_i \circ \eta_i^*(f)(X_1, \dots, X_n) = \\
&= \sum_{i=1}^{r} \sum_{j=1}^{s_i} \mathrm{res}_i \circ \eta_i^*(f)(\tau_{ij}^{(0)} X_1 \tau_{ij}^{(1)^{-1}}, \dots, \tau_{ij}^{(n-1)} X_n \tau_{ij}^{(n)^{-1}}) \\
&= \sum_{i=1}^{r} \sum_{j=1}^{s} f(\eta_i^{-1} \tau_{ij}^{(0)} X_1 (\eta_i^{-1} \tau_{ij}^{(1)})^{-1}, \dots, \eta_i^{-1} \tau_{ij}^{(n-1)} X_n (\eta_i^{-1} \tau_{ij}^{(n)})^{-1}) \\
&= \mathrm{res} \circ \mathrm{cores}(f)(X_1, \dots, X_n)
\end{aligned}$$

which is what we needed.

Lemma 3. *The following is a commutative diagram of rings and ring homomorphisms:*

$$\begin{array}{ccc} k_*(L) & \xrightarrow{h_L} & H^*(L) \\ \uparrow{\scriptstyle \mathrm{res}} & & \uparrow{\scriptstyle \mathrm{res}} \\ k_*(F) & \xrightarrow{h_F} & H^*(F) \end{array}$$

In particular if we put any $n>0$ in place of $*$ we get a commutative diagram of abelian groups.

Proof. Since $k_*(F)$ is generated as a ring by $k_1(F)$ it is enough to check the diagram for $n=1$. $h_F: k_1(F) \cong F^{\cdot}/(F^{\cdot})^2 \to H^1(F) \cong \mathrm{Hom}(G_F; Z/2Z)$ is defined for any $\alpha \in F^{\cdot}$ and $\sigma \in G_F$ by $h_F(\alpha)(\sigma) = \sigma(\sqrt{\alpha})(\sqrt{\alpha})^{-1}$ where $\sqrt{\alpha}$ is any root of $x^2-\alpha$ in $\bar{F}$. A straightforward check shows that the diagram commutes for $n=1$ and the rest follows as indicated.

Apply Lemma 3 to all the extensions L_i over E for $i=1,\ldots,r$. Then take

$$\mathrm{res} = \prod_{i=1}^{r} \mathrm{res}_i : H^n(E) \to \prod_{i=1}^{r} H^n(L_i)$$

and

$$h_{\tilde{L}} = \prod_{i=1}^{r} h_{L_i} : \prod_{i=1}^{r} k_n(L_i) \to \prod_{i=1}^{r} H^n(L_i).$$

We then get the following additional lemma:

Lemma 4. *For any $n>0$*

$$\begin{array}{ccc} k_n(\tilde{L}) & \xrightarrow{h_{\tilde{L}}} & H^n(\tilde{L}) \\ \uparrow{\scriptstyle \mathrm{res}} & & \uparrow{\scriptstyle \mathrm{res}} \\ k_n(E) & \xrightarrow{h_E} & H^n(E) \end{array}$$

is a commutative diagram.

Proof of Theorem 1. We now restrict our attention to the case where E/F is Galois of odd degree. We choose as our extension L the extension F' constructed in Section 1. We can construct a cube whose vertices are groups and whose edges are group homomorphisms in such a way that diagram (1) of Theorem 1 is one of the faces. The four diagrams of Lemmas 1–4 will provide four other faces. The sixth face is

$$\begin{array}{ccc} \prod_{i=1}^{r} k_n(L_i) & \xrightarrow{\prod_{i=1}^{r} h_{L_i}} & \prod_{i=1}^{r} H^n(L_i) \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle \mathrm{cores}} \\ k_n(L) & \xrightarrow{h_L} & H^n(L) \end{array} \tag{3}$$

where N is the sum in $k_n(L)$ of the transfer maps N_i and cores is the sum in $H^n(L)$ of cores_i. Since $L = F'$ is a union of odd degree extensions of F it follows that $\mathrm{res}: H^n(F) \to H^n(L)$ is injective (Property 4). The commutativity of diagram (1) will therefore follow from the commutativity of (3) together with the results of Lemma 1–4. The extension E/F being Galois of odd degree it therefore follows that L_i/L is Galois of odd degree. Property 2 of Section 1 says that any nontrivial finite extension of $L = F'$ must be of 2-power order. Therefore $[L_i : L] = 1$ for all $1 \le i \le r$. The commutativity of (3) is then immediate.

Remarks. If we assume that for any finite separable extension $F(\alpha)/F$ the map N_α is independent of α and is also functorial for a tower of extensions, then Theorem 1 is valid for any finite separable extension. The observations are as follows. We have already seen that the commutativity of diagram (1) follows from that of diagram (3). By Property 2 if $L = F'$ then each L_i must be a 2-power separable extension of L. By functoriality and Property 3 we need only consider the case where each L_i is a quadratic extension of L. We are therefore reduced to proving the commutativity of

$$\begin{array}{ccc} k_n(E) & \xrightarrow{h_E} & H^n(E) \\ \Big\downarrow{\scriptstyle N} & & \Big\downarrow{\scriptstyle \mathrm{cores}} \\ k_n(F) & \xrightarrow{h_F} & H^n(F) \end{array} \tag{4}$$

where $[E:F] = 2$. By [1, Corollary 5.3], $k_*(E)$ is generated as a $k_*(F)$-module by $k_1(E)$. We also know that N is a $k_*(F)$-module homomorphism ($\mathrm{res}: k_*(F) \to k_*(E)$ makes $k_*(E)$ into a $k_*(F)$-module). Finally $N: k_1(E) \cong E^\cdot/(E^\cdot)^2 \to k_1(F) \cong F^\cdot/(F^\cdot)^2$ is the classical norm map. Putting these facts together implies that (4) is a commutative diagram. The problem that remains is to define a cannonical and functorial transfer map for any finite separable extension E over F.

4.

We now turn our attention to the second theorem described in the introduction. Let $\hat{I}(F) = \ker\{\dim : \hat{W}(F) \to Z\}$, the augmentation ideal, and let its n-th power be denoted by $\hat{I}^n(F)$. Delzant [2] and Milnor [6] have generalized the classical invariants dimension, discriminant and the Hasse invariant to give an infinite sequence of invariants w_n, $n = 0, 1, 2, \ldots$. These Stiefel–Whitney invariants define maps (not necessarily homomorphisms) from $\hat{W}(F)$ to $k_n(F)$ or $H^n(F)$ depending on whether one follows Milnor's or Delzant's definition. The connection is that for $q \in \hat{W}(F)$, $w_n(q) \in H^n(F)$ of Delzant equals $h_F(w_n(q))$, where $w_n(q) \in k_n(F)$ is as defined by Milnor. Recall now the definition of $k_\pi(F)$ as given in Section 2. Define $H^\pi(F)$ analogously. The w_i induce a homomorphism $w: \hat{W}(F) \to k_\pi(F)^\cdot$ (or $H^\pi(F)^\cdot$).

$w = w_0 + w_1 + w_2 + \cdots$ is called the total Stiefel–Whitney class. The homomorphism is from the additive group $\hat{W}(F)$ to the multiplicative group of units in $k_\pi(F)$ (or $H^\pi(F)$). From now on when we use the w_i we will view them as maps from $\hat{W}(F)$ to $H^i(F)$. Note that although w_i is not a homomorphism, if $t = 2^{n-1}$, $n \geq 1$, then by [6, Lemma 3.2], $w_t : \hat{I}^n(F) \to H^t(F)$ is a group homomorphism.

Theorem 2. *If $t = 2^{n-1}$ where $n \geq 1$, then*

$$\begin{tikzcd} \hat{I}^n(E) \arrow[r, "w_t"] \arrow[d, "T"] & H^t(E) \arrow[d, "\text{cores}"] \\ \hat{I}^n(F) \arrow[r, "w_t"] & H^t(F) \end{tikzcd}$$

is a commutative diagram of groups and group homomorphisms.

The proof of Theorem 2 will follow as a corollary to the main theorem of [8]. Besides corestriction there exists a multiplicative transfer map $\mathcal{N} : H^\pi(E)^\bullet \to H^\pi(F)^\bullet$. $\mathcal{N}$ is a multiplicative group homomorphism from the group of units of $H^\pi(E)$ to the group of units of $H^\pi(F)$ [8]. The map $\mathcal{N}$ is defined as a direct limit of $\mathcal{N}_i : H^\pi(\mathrm{Gal}(E_i/E)) \to H^\pi(\mathrm{Gal}(F_i/F))$ where $\{E_i\}_{i \in \Lambda}$ is the set of all finite Galois extensions of E contained in $\bar{F}$. Recall that $H^\pi(E) \cong \varinjlim_{i \in \Lambda} H^\pi(\mathrm{Gal}(E_i/E))$. For each $i \in \Lambda$, $\mathrm{Gal}(E_i/E)$ is a subgroup of the finite group $\mathrm{Gal}(E_i/F)$, of index $l = [E : F]$. For a finite group the definition of $\mathcal{N}_i$ goes back to Evens [3].

Theorem (main theorem of [8]). *Let E be a finite Galois extension of F. If $[E : F]$ is odd then the diagram*

$$\begin{tikzcd} \hat{W}(E) \arrow[r, "w"] \arrow[d, "T"] & H^\pi(E)^\bullet \arrow[d, "\mathcal{N}"] \\ \hat{W}(F) \arrow[r, "w"] & H^\pi(F)^\bullet \end{tikzcd} \tag{5}$$

is commutative. If $[E : F]$ is even, then the diagram commutes modulo the kernel of the restriction map, $\mathrm{res} : H^\pi(F) \to H^\pi(E)$.

We now introduce a lemma which will relate $\mathcal{N}$ to the corestriction homomorphism.

Lemma. *Let G be a profinite group and let G' be a subgroup of G of finite index. Suppose $\gamma \in H^\pi(G)$ is of the form $1 + \alpha + \alpha_1 + \alpha_2 + \cdots$ where $\alpha \in H^n(G')$ and for $i = 1, 2, 3, \ldots$, $\alpha_i \in H^i(G')$ with $i > n$. Then*

$$\mathcal{N}(\gamma) = 1 + \mathrm{cores}(\gamma) + \beta_1 + \beta_2 + \cdots$$

where $\beta_j \in H^j(G)$ *and* $j > n$ *for* $j = 1, 2, \ldots$.

Proof.

Since $\mathcal{N}$ is defined as a direct limit, as noted, it is enough to prove the lemma for the case where G is a finite group. We must also note of course that for a profinite group corestriction can be defined as a direct limit of the corestriction maps on each finite level. For the case of a finite group the lemma becomes a slight generalization of Theorem 1 in [3].

Let $1 = \tau_1, \ldots, \tau_l$ be a left transversal of G' in G. This set induces a group homomorphism $\Phi : G \to S_l \tilde{X} (G')^l$, the semi-direct product of the symmetric group S_l and $(G')^l$. The action of S_l is by permuting the entries of $(G')^l$ [3, p. 55]. Let

$$W = \cdots \to W_i \to W_{i-1} \to \cdots \to W_0 \xrightarrow{\varepsilon} Z \to 0$$

be an S_l-projective resolution of Z and let

$$X = \cdots \to X_i \to X_{i-1} \to \cdots \to X_0 \xrightarrow{\mu} Z \to 0$$

be a G'-projective resolution of Z. Then $W \otimes X^l$ (tensor product of resolutions) is an $S_l \tilde{X} (G')^l$ projective resolution of Z and by the map Φ also a G projective resolution of Z.

Define $\eta : Z \to Z/2Z$ by $\eta(1) = 1 \pmod 2$ then $\mu' = \eta \circ \mu : X_0 \to Z/2Z$ will represent $1 \in H^0(G')$. Let $f : X_n \to Z/2Z$ represent $\alpha \in H^n(G')$ and for $i \geq 1$ let $f_i : X_i \to Z/2Z$ represent $\alpha_i \in H^i(G')$. The maps f, f_i may also be viewed as maps on the entire chain X by setting them equal to the zero map in all but the n-th, i-th component, respectively. By [3, Section 5.5], $\mathcal{N}(\gamma)$ is represented in $\mathrm{Hom}_G(W \otimes X^l; Z/2Z)$ by $\mathcal{N}(\mu' + f + f_1 + \cdots)$ where for any $g : X \to Z/2Z$

$$\mathcal{N}(g)(w_0 \otimes x_1 \otimes \cdots \otimes x_l) = \varepsilon'(w_0) g(x_1) \cdots g(x_l), \quad x_i \in X.$$

Since for $i \geq 1$ $f_i : X_j \to Z/2Z$ is zero for any $j \leq n$ it follows that if we wish to restrict $\mathcal{N}(\gamma)$ to $\mathrm{Hom}_G((W \otimes X^l)_n; Z/2Z)$ we may use as a representative $\mathcal{N}(\mu' + f)$. By [3, first paragraph on p. 64], $\mathcal{N}(\mu' + f)$ represents $\mathrm{cores}(\alpha)$ in $H^n(G)$. This completes the proof of the lemma.

Proof of Theorem 2. Let us apply diagram (5) to an element $q \in \hat{I}^n(E) \leq \hat{W}(E)$. By [6, Lemma 3.2], $w(q) = 1 + w_t(q) +$ higher terms. By the lemma it follows that $\mathcal{N}(w(q)) = 1 + \mathrm{cores}(w_t(q)) +$ higher terms. We have $T(q) \in \hat{I}^n(F) \leq \hat{W}(F)$ and $w(T(q)) = 1 + w_t(T(q)) +$ higher terms. Therefore it follows that $\mathrm{cores}(w_t(q)) = w_t(T(q))$ as claimed.

Remarks. (1) If E/F is Galois of even degree then as in the main theorem of [8] we have $\mathrm{cores} \circ w_t = w_t \circ T$ on $\hat{I}^n(E)$, modulo the kernel of $\mathrm{res} : H^t(F) \to H^t(E)$.

(2) Whenever $h_F : k_\pi(F) \to H^\pi(F)$ is injective (also $h_F : k_\pi(E) \to H^\pi(E)$) we can

define $\mathcal{N} : k_\pi(E)^{\cdot} \to k_\pi(F)^{\cdot}$. As previously noted h_F is known to be bijective for many cases. We have also shown (Theorem 1) that if we identify $k_\pi(E)$ as a subring of $H^\pi(E)$ via h_E then the Bass–Tate transfer 'is' corestriction. Theorem 2 then implies that $N \circ W_t = T \circ W_t$ on $\hat{I}^n(E)$, where N is the Bass–Tate transfer. It would be extremely enlightening to realize $\mathcal{N} : k_\pi(E)^{\cdot} \to k_\pi(F)^{\cdot}$ from a purely K-theoretic perspective.

(3) By Tate's result [10], if $T_2 : k_2(E) \to k_2(F)$ is the transfer map defined by Milnor in [1] then $\mathrm{cores} \circ h_E = h_F \circ T_2$ on $k_2(E)$. If h_F is injective then it follows from Theorem 1 that $T_2 = N$, where N is the Bass–Tate transfer on $k_2(E)$. This is extremely interesting considering the divergence of their respective definitions.

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